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# Transverse motion of an elastic sphere in a shear field

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The forces acting on an elastic particle suspended in a shear field, and moving relative to it, are found for the case in which there are small deformations from an initially spherical shape. The deformation is the result of the viscous stresses acting on the particle. Of principal interest is that there is a component of the force perpendicular to the free-stream direction, so that the particle will migrate across the undisturbed streamlines.

## 1. Introduction

There has been a wide range of experimental and theoretical studies of the behaviour of particles suspended in a fluid flowing through a tube. Much of this work has been motivated by observations of blood flow which disclosed that the red blood cells appeared to congregate near the axis of a blood vessel (see Whitmore 1968, for example). The fluid mechanical forces causing this migration, and other migration phenomena (Segre' & Silberberg 1962), have received considerable attention. Brenner (1966) has reviewed the theories and relevant experiments in this area. A more recent review by Cox & Mason (1971) considers additional work in this area as well as rheological effects.

The migration of solid spherical particles has been found to be due to inertial effects, there being no migration at very small Reynolds number. Theoretical analyses of these inertial forces have been given by Rubinow & Keller (1961) and Saffman (1965).

When the particles are deformable, migration does occur at low Reynolds number as has been observed for liquid drops and flexible fibres (see Cox & Mason 1971). Chaffey, Brenner & Mason (1965) considered the interaction of drop deformation and wall effects to show migration. More recently Haber & Hetsroni (1971) found a side force on a liquid drop suspended in a Poiseuille flow without considering wall effects.

The purpose of this work is to show that an elastic particle experiences a side force due to deformation and that this force is directed toward the axis of a tube under conditions applicable to blood flow. An elastic particle is a significant improvement over a liquid drop as a model of blood cells since it requires the

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no-slip condition at the particle surface with the particle accommodating itself to the surface stresses. This is in marked contrast to liquid drops, in which velocity and shear stress are continuous across the interface. The red blood cell is thought to consist of a membrane containing a highly concentrated solution in which some structure may be present.

## 2. Mathematical formulation

The specific problem being considered here consists of an elastic particle, spherical at rest with radius $a$, which is contained in a simple shear flow with velocity

$$\mathbf{U}_\infty = (U_\infty + \beta y)\,\hat{\mathbf{x}} \tag{1}$$

relative to the particle. The particle is rotating with an angular velocity $\omega\hat{\mathbf{z}}$. The caret here denotes unit vectors in the specified direction.

The motion is assumed to be sufficiently slow that inertial terms can be neglected. The equations of motion and continuity for the fluid then are the familiar equations of slow viscous flow

$$-\nabla p + \mu\nabla^2\mathbf{U} = 0, \quad \nabla . \mathbf{U} = 0, \tag{2), (3}$$

where $p$ is the pressure, $\mu$ the viscosity and $\mathbf{U}$ the velocity field. The boundary conditions on the flow are

$$\mathbf{U} \rightarrow \mathbf{U}_\infty \quad \text{at infinity} \tag{4}$$

and

$$\mathbf{U} = \mathbf{U}_s \tag{5}$$

at the surface of the sphere, where $\mathbf{U}_s$ is the surface velocity of the elastic material.

To describe the deformation of the particle, it is assumed that the elastic displacements are small so that linear elasticity theory applies. It is also assumed that the speed of stress waves is large so that the particle shape is dictated by the stress field at any instant. The material of the sphere is taken as homogeneous and isotropic and characterized by the stress–strain relation

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij}, \tag{6}$$

where $\lambda$ and $G$ are Lamé's constants and the strains $e_{ij}$ are given in terms of the displacement field $\mathbf{S}^*$ by

$$e_{ij} = \frac{\partial S_i^*}{\partial x_j} + \frac{\partial S_j^*}{\partial x_i}. \tag{7}$$

The displacement is governed by the Navier equation

$$(\lambda + G)\,\nabla(\nabla . \mathbf{S}^*) + G\nabla^2\mathbf{S}^* + \mathbf{F} = 0, \tag{8}$$

where $\mathbf{F}$ is the body force. The boundary conditions on the displacements will be given in terms of the stress at the particle surface.

## 3. Solution

The solution to this problem involves the simultaneous determination of the flow field and the shape of the deformed particle. For small deformations this can be accomplished by a perturbation procedure in which the expansion is with respect to the ratio of the deformation to the original sphere radius. This procedure is similar to that used by Hyman & Skalak (1970) and Haber & Hetsroni (1971) for liquid drops, in which case the magnitude of the deformation is proportional to a dimensionless quantity measuring the ratio of the viscous force to the surface-tension force. In the present case the magnitude of the deformation is proportional to the ratio of the viscous stress to the elastic stress. This ratio occurs in more than one form as will be subsequently seen and is therefore not readily designated by a single dimensionless group. None the less the actual expansion parameter is the deformation for both the liquid drop and the present case of a solid elastic sphere. It will be shown below that the zero-order solution here is the same as that for a spherical particle in the same sense that the zero-order solution for the case of a liquid drop is that appropriate for a spherical liquid drop. A significant feature of the present case, however, is that in the absence of deformation a solid elastic particle is indistinguishable from a rigid particle. This of course is not the case for an undeformed liquid drop (unless the viscosity of the drop liquid is very large).

### 3.1. *Perturbation analysis*

In the absence of deformation the elastic particle under consideration here is spherical. It is therefore natural in the case of small deformations to express the contour of the particle as

$$\mathbf{r}_0 = a\hat{\mathbf{e}}_r + \epsilon\mathbf{S}(a, \theta, \phi). \tag{9}$$

Here $\epsilon\mathbf{S}$ is equal to the actual displacement $\mathbf{S}^*$. For small deformations we require $|\mathbf{S}|$ to be of the same order as $a$ and $\epsilon$ to be small. It follows from (9) that

$$\frac{r_0^2}{a^2} = \left(1+\frac{\epsilon S_r}{a}\right)^2 + \left(\frac{\epsilon S_\theta}{a}\right)^2 + \left(\frac{\epsilon S_\phi}{a}\right)^2 \tag{10}$$

and finally that

$$r_0 = a + \epsilon S_r \tag{11}$$

to first order in the deformation.

The velocity field sought can be represented as the field which would occur in the absence of any deformation plus small changes:

$$\mathbf{U} = \mathbf{U}^{(0)} + \epsilon\mathbf{U}^{(1)} + \dots. \tag{12}$$

In order to evaluate $\mathbf{U}$ at the surface $\mathbf{r}_0$ it is convenient to expand the velocity at the particle surface in terms of its values at the spherical surface $r = a$:

$$\mathbf{U}(r_0) = \mathbf{U}^{(0)}(a, \theta, \phi) + \epsilon\mathbf{U}^{(1)}(a, \theta, \phi) + \epsilon\frac{\partial\mathbf{U}^{(0)}}{\partial r}(a, \theta, \phi)\, S_r. \tag{13}$$

In this equation only terms to first order in $\epsilon$ are retained.

The surface velocity $\mathbf{U}_s$ of the particle is approximately that of pure rotation but a correction is necessary since a point on the contour is moving tangentially to the contour rather than in the direction $\boldsymbol{\omega} \times \mathbf{r}_0$. Therefore $\mathbf{U}_s$ can be written as

$$\mathbf{U}_s = \boldsymbol{\omega} \times \mathbf{r}_0 + \epsilon\mathbf{f}, \tag{14}$$

where the vector $\mathbf{f}$ ($\mathbf{f} = f\hat{\mathbf{n}}$) is determined by the requirement that

$$\hat{\mathbf{n}} . \mathbf{U}_s = 0. \tag{15}$$

Here $\hat{\mathbf{n}}$ is the unit normal to the surface of the deformed sphere. To the order being considered here

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_r - \frac{\epsilon}{a}\frac{\partial S_r}{\partial \theta}\hat{\mathbf{e}}_\theta - \frac{\epsilon}{a\sin\theta}\frac{\partial S_r}{\partial \phi}\hat{\mathbf{e}}_\phi. \tag{16}$$

It follows that

$$\mathbf{f} = (\boldsymbol{\omega}\times\hat{\mathbf{e}}_r)\cdot\left(\frac{\partial S_r}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{\sin\theta}\frac{\partial S_r}{\partial\phi}\hat{\mathbf{e}}_\phi\right)\hat{\mathbf{e}}_r. \tag{17}$$

The boundary condition (5) can now be written (to order $\epsilon$) in the form

$$\mathbf{U}^{(0)}(a,\theta,\phi) + \epsilon\mathbf{U}^{(1)}(a,\theta,\phi) + \epsilon\frac{\partial\mathbf{U}^{(0)}}{\partial r}(a,\theta,\phi)\,S_r = a\boldsymbol{\omega}\times\hat{\mathbf{e}}_r + \epsilon S_r\boldsymbol{\omega}\times\hat{\mathbf{e}}_r + \epsilon\mathbf{f}. \tag{18}$$

The boundary condition at infinity (equation (4)) is satisfied by requiring

$$\mathbf{U}^{(0)} \to \mathbf{U}_\infty, \quad \mathbf{U}^{(1)} \to 0 \quad \text{at infinity.} \tag{19a, b}$$

### 3.2. *Zero-order solution*

To zero order in $\epsilon$ the problem reduces to finding the velocity field which satisfies (19$a$), the zero-order condition at the sphere surface from (18),

$$\mathbf{U}^{(0)} = a\boldsymbol{\omega}\times\hat{\mathbf{e}}_r \quad \text{at} \quad r = a, \tag{20}$$

and the equations of motion given by (2) and (3). This problem can be solved using Lamb's (1945, p. 594) general solution and has been given before (Saffman 1965; Tam 1966). The solution in terms of spherical co-ordinates $(r,\theta,\phi)$ is

$$\begin{aligned}\mathbf{U}^{(0)} = \hat{\mathbf{e}}_r&\left\{U_\infty\left[1-\frac{3a}{2r}+\frac{a^3}{2r^3}\right]\cos\theta+\beta\left[-\frac{r}{3}+\frac{5a^3}{6r^2}-\frac{a^5}{2r^4}\right](-3\sin\theta\cos\theta\cos\phi)\right\}\\ &+\hat{\mathbf{e}}_\theta\left\{U_\infty\left[-1+\frac{3a}{4r}+\frac{a^3}{4r^3}\right]\sin\theta+\beta\left[\frac{r}{5}-\frac{a^5}{5r^4}\right](5\cos^2\theta-1)\cos\phi\right.\\ &\left.+\beta\left[\frac{3a^5}{10r^2}+\frac{a^3}{2r^2}-\frac{4r}{5}\right]\cos\phi+\frac{a^3\omega}{r^2}\cos\phi\right\}\\ &+\hat{\mathbf{e}}_\phi\left\{\beta\left[\frac{a^5}{2r^4}-\frac{a^3}{2r^2}\right]\cos\theta\sin\phi-\frac{a^3\omega}{r^2}\cos\theta\sin\phi\right\},\end{aligned} \tag{21}$$

$$p^{(0)} = \frac{-3\mu a U_\infty}{2r^2}\cos\theta - \frac{5\beta\mu a^2}{r^2}\sin\theta\cos\theta\cos\phi. \tag{22}$$

Here $x$ is the polar axis and $\phi$ is measured from the $z$ axis.

The stresses acting on the surface of the sphere owing to the above flow are

$$\tau_{rr} = \frac{3\mu U_\infty}{2a}P_1 + \frac{5\beta\mu}{3}P_2^1\cos\phi, \tag{23a}$$

$$\tau_{r\theta} = \frac{1}{\sin\theta}\left\{\frac{-\mu U_\infty}{a} + \frac{\mu U_\infty}{a}P_2 + 3\mu(\beta+\omega)P_1^1\cos\phi - \frac{2\beta\mu}{3}P_3^1\cos\phi\right\}, \tag{23b}$$

$$\tau_{r\phi} = \frac{1}{\sin\theta}\left\{\frac{\beta\mu}{3}P_2^1\sin\phi - \mu\omega P_2^1\sin\phi\right\}, \tag{23c}$$

where $P_n^m$ are associated Legendre functions of argument $\cos\theta$.

The drag on the sphere due to this zero-order stress field is easily computed. The result is

$$\mathbf{D} = 6\pi\mu a U_\infty \hat{\mathbf{x}} \tag{24}$$

and the torque is

$$\mathbf{T} = -8\pi\mu a^2(\omega + \tfrac{1}{2}\beta)\hat{\mathbf{z}}. \tag{25}$$

This drag and torque must be balanced by body forces acting on the sphere. The drag may be considered to be balanced by a constant body force per unit volume

$$\mathbf{f}_1 = \frac{-9\mu U_\infty}{2a^2}\hat{\mathbf{x}}. \tag{26}$$

The torque is balanced by a body force of the form

$$\mathbf{f}_2 = A\hat{\mathbf{z}} \times \mathbf{r}, \tag{27}$$

where $A$ is determined by

$$\int A(\hat{\mathbf{z}} \times \mathbf{r}) \times \mathbf{r}\, dV = -8\pi\mu a^2(\omega + \tfrac{1}{2}\beta)\hat{\mathbf{z}}; \tag{28}$$

$$A = (15\mu/a^2)(\tfrac{1}{2}\beta + \omega). \tag{29}$$

### 3.3. *Deformation field*

The first-order displacement field ($\mathbf{S}^*$) of the particle can be computed from (8) with

$$\mathbf{F} = -\frac{9\mu U_\infty}{2a^2}\hat{\mathbf{x}} + \frac{15\mu}{a^2}(\tfrac{1}{2}\beta + \omega)\hat{\mathbf{z}} \times \mathbf{r} \tag{30}$$

and the surface stresses (23). The particular solution to (8) corresponding to the above body force is

$$\begin{aligned}
\mathbf{S}_p^* = \hat{\mathbf{e}}_r &\left\{ \frac{9\mu U_\infty r^2}{4a^2 G[2 + (\lambda/G)]}[\tfrac{2}{5}P_3 + \tfrac{3}{5}P_1] + \frac{5\mu}{2a^2 G}(\tfrac{1}{2}\beta + \omega) r^3 [-\tfrac{1}{420}P_4^3 \cos 3\phi + \tfrac{1}{10}P_4^1 \cos\phi] \right\} \\
&+ \frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\left\{ \frac{9\mu U_\infty r^2}{4a^2 G[2 + (\lambda/G)]}[\tfrac{2}{15}P_3^1 + \tfrac{1}{5}P_1^1]\sin\phi - \frac{5\mu}{2a^2 G}(\tfrac{1}{2}\beta + \omega) r^3 \right. \\
&\left. \times [\tfrac{1}{1890}P_5^3 \cos 3\phi - \tfrac{2}{135}P_3^3 \cos 3\phi - \tfrac{2}{45}P_5^1 \cos\phi + \tfrac{2}{45}P_3^1 \cos\phi - \tfrac{3}{5}P_1^1 \cos\phi] \right\} \\
&+ \frac{\hat{\mathbf{e}}_\phi}{\sin\theta}\left\{ \frac{5\mu}{2a^2 G}(\tfrac{1}{2}\beta + \omega) r^3 [-\tfrac{2}{35}P_4^1 - \tfrac{1}{7}P_2^1]\sin\phi \right\}.
\end{aligned} \tag{31}$$

A general solution to (8) similar to Lamb's (1945) general solution to the Stokes equation is given in appendix A.

The surface stresses corresponding to the displacements of (31) are

$$\begin{aligned}
\tau_{rr} = &\frac{18\mu U_\infty}{5a[2 + (\lambda/G)]}P_3 + \frac{9\mu U_\infty}{a[2 + (\lambda/G)]}\frac{6 + 5\lambda/G}{10}P_1 \\
&+ 15\mu a(\tfrac{1}{2}\beta + \omega)[-\tfrac{1}{420}P_4^3 \cos 3\phi + \tfrac{1}{10}P_4^1 \cos\phi],
\end{aligned} \tag{32a}$$

$$\begin{aligned}
\tau_{r\theta} = \frac{1}{\sin\theta} &\left\{ \frac{9\mu U_\infty}{a[2 + (\lambda/G)]}[\tfrac{8}{35}P_4 - \tfrac{2}{21}P_2 - \tfrac{2}{15}] + \frac{15\mu}{2}(\tfrac{1}{2}\beta + \omega) \right. \\
&\left. \times [-\tfrac{1}{945}P_5^3 \cos 3\phi + \tfrac{7}{540}P_3^3 \cos 3\phi + \tfrac{4}{45}P_5^1 \cos\phi - \tfrac{11}{10}P_3^1 \cos\phi + \tfrac{2}{5}P_1^1 \cos\phi] \right\},
\end{aligned} \tag{32b}$$

$$\tau_{r\phi} = \frac{1}{\sin\theta}\left\{ \frac{15\mu}{2a}(\tfrac{1}{2}\beta + \omega)[\tfrac{1}{420}P_4^3 \sin 3\phi - \tfrac{1}{14}P_4^1 \sin\phi - \tfrac{2}{21}P_2^1 \sin\phi] \right\}. \tag{32c}$$

The unknown coefficients $A_{1m}$, $B_{1m}$ and $C_{1m}$ of the homogeneous solution can now be determined by equating the elastic stresses (equations (32) and stress formulae derived from (A 1)) to the viscous stresses of (23). Since the functions $P_n^m \cos m\phi$ are mutually orthogonal the coefficients of each of these functions can be set equal to zero. In this way sets of linear algebraic equations for the coefficients are obtained. It will be shown subsequently that only some of the coefficients of the displacement field need be found in order to compute the next-order forces on the particle. These coefficients are $B_{10}^1$, $B_{30}^1$, $C_{30}^1$, $B_{21}^1$ and $C_{21}^1$. Explicit expressions for these coefficients are given in appendix B.

Once the coefficients of the homogeneous solution are determined the equation of the contour of the particle is obtained by adding $\epsilon S_r(S_r^*)$ to $a$ as in (11). In order for the deformation to be small the coefficients in (31) and appendix B must be no larger than $\epsilon$. Thus $\epsilon$ is related to the ratio of the viscous forces to the elastic forces. In the general problem there are three characteristic velocities, $U_\infty$, $a\omega$ and $a\beta$, so that this ratio takes the several forms $\mu U_\infty/G$, $\mu a\beta/G$ and $\mu a(\omega+\frac{1}{2}\beta)/G$. The ratio $\lambda/G$ is needed to complete the specification of the problem.

### 3.4. *Drag on the deformed particle*

The boundary condition on $\mathbf{U}^{(1)}$ for the next-order flow problem is determined by equating terms multiplied by $\epsilon$ in (18). This takes the form

$$\mathbf{U}^{(1)} = -\frac{\partial \mathbf{U}^{(0)}}{\partial r} S_r + S_r \boldsymbol{\omega} \times \hat{\mathbf{e}}_r + \mathbf{f}, \tag{33}$$

where all terms are evaluated at $r = a$. Also from (19*b*)

$$\mathbf{U}^{(1)} \to 0 \quad \text{at infinity.} \tag{34}$$

The solution to this problem could now be obtained by again employing Lamb's (1945) general solution. However, the objective of this study is to determine the transverse component $D_y$ of the drift force, and this can be accomplished without finding $\mathbf{U}^{(1)}$ explicitly by using the relationship (Tam 1969)

$$\mathbf{D} = \frac{3}{2}\frac{\mu}{a}\int \mathbf{U}^{(1)}\, dS, \tag{35}$$

where the integral is over the sphere surface. The advantage here is that the surface values of $\mathbf{U}^{(1)}$ are already known from (33). The integration indicated in (35) is a somewhat tedious procedure involving various integrals of products of the spherical harmonics. Furthermore, portions of the above problem have been previously considered. Therefore the relevant results will simply be quoted here.

Tam (1966) considered the flow $\mathbf{U}_\infty$ past a distorted non-rotating sphere and found the velocity field explicitly as well as the total drag. The transverse component from his result is

$$D_y = 12\pi\left[\frac{3\mu a}{20} U_\infty \epsilon_{21} - \mu a^2 \beta(\tfrac{5}{7}\epsilon_{32} - \tfrac{1}{28}\epsilon_{30})\right], \tag{36}$$

where the quantities $\epsilon_{1m}$ are the coefficients in the expansion

$$r_0 = a\left(1 + \sum_{l=0}^{\infty}\sum_{m=0}^{l}[\epsilon_{lm}P_l^m \cos m\phi + \delta_{lm} P_l^m \sin m\phi]\right). \tag{37}$$

The above expression contains Brenner's (1964) result for uniform flow past a distorted sphere as a special case (see also Happel & Brenner (1965, p. 207), which gives general procedures for effecting these solutions).

The pure rotation of a distorted sphere in an otherwise quiescent fluid was also considered by Brenner (1964). Writing that result in the present notation yields

$$D_y = -6\pi\mu a^2 \omega \epsilon_{10}. \tag{38}$$

The remaining problem in the boundary condition (33) is to find the forces associated with the radial flow **f**. The treatment here is similar to the above analysis and the result is

$$D_y = -4\pi\mu a^2 \omega \epsilon_{10} - 36\pi\mu a^2 \omega \sum_{n=3,5}^{\infty} \epsilon_{n2}. \tag{39}$$

Finally, then, the total transverse force is obtained by adding the separate results of (36), (38) and (39). In terms of the present displacement field this result reduces to

$$D_y = 4\pi\left[\frac{9\mu a}{20} U_\infty C_{21}^1 - 5\mu a^2 \omega \left(\frac{27}{20}\frac{\mu U_\infty}{aG[2+(\lambda/G)]} + aB_{10}^1\right)\right], \tag{40}$$

where the coefficients are given in appendix B. It is worth noting here that while this phase of the problem was considered as a linear superposition of its component parts that is not possible for the problem as a whole. The deformation of the sphere is dependent on the uniform flow, shear and rotation. In the first-order problem the combined deformation must be used in the separate solutions outlined above. In this way the different flows become coupled.

## 4. Conclusions and discussion

The principal conclusion of the work is that owing to deformation of an initially spherical, solid, elastic particle in a shear flow additional drag forces are developed. There is a component of this drag force perpendicular to the free-stream direction which would cause a freely suspended particle to drift across the streamlines of the undisturbed flow. The direction of drift is in the direction of $D_y$ (equation (40)) and depends on the relative velocity $U_\infty$ and the sign of the shear $\beta$ and the rotation $\omega$.

In applying this result to the motion of a deformable particle suspended in a Poiseuille flow (the case of interest for biological applications) it is necessary to determine the appropriate values of $U_\infty$, $\beta$ and $\omega$. This aspect of the problem has been considered previously by Saffman (1965), Rubinow & Keller (1961) and others. It seems reasonable to use the local shear rate for $\beta$ and the condition of zero net torque to give

$$\omega = -\tfrac{1}{2}\beta. \tag{41}$$

The relative velocity $U_\infty$ has usually been taken to be

$$U_\infty = \tfrac{2}{3}(a/R_0)^2 U_0, \tag{42}$$

where $R_0$ is the tube radius and $U_0$ the centre-line velocity. The above result follows from Faxen's laws (Happel & Brenner 1965, p. 316) and is strictly applicable only to a rigid sphere, whether rotating or not. This is the appropriate velocity here, however, since the required $U_\infty$ enters from the zero-order solution, in which the particle is undeformed and therefore indistinguishable from a rigid sphere. This situation must not be confused with the equivalent expression for a liquid drop given by Hetsroni, Haber & Wacholder (1970), in which case

$$U_\infty = \frac{2\sigma}{3\sigma+2}\left(\frac{a}{R_0}\right)^2 U_0, \tag{43}$$

where $\sigma$ is the ratio of the viscosity of the drop to the viscosity of the suspending medium. In this case, even though the drop remains spherical the material of the drop has a significant effect on the relative velocity. This effect arises since in the liquid drop there is internal circulation which results in the equivalent of a slip boundary condition at the interface between the drop and the suspending fluid.

The possibility does exist, however, that the centre-line velocity difference given above should not be used but rather some average velocity. This has been considered by Reppetti & Leonard (1964) but without adequate theoretical justification. In addition, it should be noted that a wall effect also acts to decrease the velocity of a particle relative to that of the fluid.

From (41) and (42), the direction of migration predicted by (40) is towards the tube axis. This conclusion does not strictly depend on the magnitude of the velocity given in (42) but only on the fact that the particle has a lower velocity than the undisturbed fluid velocity through its centre. Thus any uncertainty in using (42) does not significantly alter the result.

The direction of the migration obtained here is the same as that predicted by Saffman (1965) but for totally different reasons. Saffman's (1965) result depends on inertial effects since a rigid sphere has been shown not to experience a side force in slow viscous flow. In the present work inertial effects are completely absent, the side force resulting from the perturbation in the flow field caused in turn by elastic deformation.

## Appendix A. Elastic distortion of a sphere

Part of this appendix is based on chapter 13 of Morse & Feshbach (1953).

*Navier equation*

$$(\lambda+G)\nabla(\nabla.\mathbf{S}^*)+G\nabla^2\mathbf{S}^* = 0.$$

A general solution of the homogeneous Navier equation (finite at $r = 0$) can be written as

$$\begin{aligned}\mathbf{S}^* = \hat{\mathbf{e}}_r\Bigg\{&\sum_{n=0}^{\infty}\sum_{m=0}^{n}[r^{n+1}P_n^m(\cos\theta)\,(B_{nm}^1\cos m\phi+B_{nm}^0\sin m\phi)]\\ &+\sum_{n=1}^{\infty}\sum_{m=0}^{n}[r^{n-1}P_n^m(\cos\theta)\,(C_{nm}^1\cos m\phi+C_{nm}^0\sin m\phi)]\Bigg\}\\ &+\frac{\hat{\mathbf{e}}_\theta}{\sin\theta}\Bigg\{\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[r^n\frac{m(2n+1)}{n(n+1)}P_n^m(\cos\theta)\,(-A_{nm}^1\sin m\phi+A_{nm}^0\cos m\phi)\right]\end{aligned}$$

$$
+\left[\frac{nr^{n+1}}{(2n+1)}\frac{\lambda(n+3)+G(n+5)}{\lambda n+G(n-2)}\left(\frac{n-m+1}{n+1}P_{n+1}^{m}-\frac{n+m}{n}P_{n-1}^{m}\right)\right.
$$

$$
\left.\times\left(B_{nm}^{1}\cos m\phi+B_{nm}^{0}\sin m\phi\right)\right]
$$

$$
+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[\frac{(n+1)\,r^{n-1}}{2n+1}\left(\frac{n-m+1}{n+1}P_{n+1}^{m}(\cos\theta)-\frac{n+m}{n}P_{n-1}^{m}(\cos\theta)\right)\right.
$$

$$
\left.\left.\times\left(C_{nm}^{1}\cos m\phi+C_{nm}^{0}\sin m\phi\right)\right]\right\}
$$

$$
+\frac{\hat{\mathbf{e}}_{\phi}}{\sin\theta}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\left(-\frac{n-m+1}{n+1}P_{n+1}^{m}(\cos\theta)+\frac{n+m}{n}P_{n-1}^{m}(\cos\theta)\right)\right.\right.
$$

$$
\left.\times\left(A_{nm}^{1}\cos m\phi+A_{nm}^{0}\sin m\phi\right)\right]r^{n}
$$

$$
+\left[\frac{mr^{n+1}}{n+1}\frac{\lambda(n+3)+G(n+5)}{\lambda n+G(n-2)}P_{n}^{m}(\cos\theta)\left(-B_{nm}^{1}\sin m\phi+B_{nm}^{0}\cos m\phi\right)\right]
$$

$$
\left.+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[\frac{mr^{n-1}}{n}P_{n}^{m}(\cos\theta)\left(-C_{nm}^{1}\sin m\phi+C_{nm}^{0}\cos m\phi\right)\right]\right\}. \qquad \text{(A 1)}
$$

## Appendix B

Writing out explicitly the formulae for the coefficients $B_{10}^1$, $C_{10}^1$, $C_{20}^1$, $B_{21}^1$ and $C_{21}^1$ yields

$$
B_{10}^{1}=\frac{3}{10}\frac{\mu U_{\infty}}{Ga^{2}}\frac{[4+5(\lambda/G)]\,[(\lambda/G)-1]}{[2+(3\lambda/G)]\,[2+(\lambda/G)]}, \qquad \text{(B 1)}
$$

$$
B_{30}^{1}=B_{21}^{1}=0, \qquad \text{(B 2)}
$$

$$
C_{30}^{1}=\frac{-9\mu U_{\infty}}{10a^{2}G[2+(\lambda/G)]}, \qquad \text{(B 3)}
$$

$$
C_{21}^{1}=-\frac{5\mu\beta}{6G}. \qquad \text{(B 4)}
$$

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